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Distortion of Twisted Orientation Patterns in Liquid Crystals by Magnetic Fields†

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Abstract—At rest between parallel plates, cholesteric liquid crystals commonly exhibit a characteristic twisted orientation pattern, the axis of twist being perpendicular to the plates. Also, this orientation pattern appears possible in nematic liquid crystals, since it is consistent with continuum theory[1] for both types of liquid crystal. This paper discusses the influence upon such a twisted orientation pattern of a magnetic field perpendicular to the plates. If one employs a free energy of the form discussed by Frank,[2] the continuum theory equations have solutions relevant to this situation. For nematic liquid crystals, provided that the twist is not too large, our analysis suggests that no distortion of the orientation pattern occurs until the magnetic field strength exceeds a critical value which varies with the amount of twist. For cholesteric liquid crystals, there seems to be two possibilities depending upon the relative magnitudes of two Frank constants. Either distortion always occurs above a critical field strength, or does so only when the distance between the plates is sufficiently small.

1. Introduction

A number of interesting experiments in liquid crystal theory employ solid boundaries and external magnetic fields as competing influences upon the orientation of the large elongated molecules which occur in these liquids. For a nematic liquid crystal at rest in a small gap between parallel plates, suitable prior treatment of the solid surfaces leads to a uniform orientation pattern either parallel or perpendicular to the plates (see for example Chatelain[3]). Application of a magnetic field to such thin films of nematic liquid crystal leads to three important experiments. Two employ a parallel orientation pattern

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with the field either normal or parallel to the plates, but always perpendicular to the molecular axes, and the third uses a perpendicular orientation pattern with the field parallel to the plates. In each case there is no appreciable distortion of the orientation pattern until the magnetic field strength exceeds a critical value, which depends upon the liquid crystal, the distance between the plates, and the arrangement employed. Continuum theory predicts such behaviour, and in this way Saupe[4] obtains estimates for three parameters in the Frank energy.[2] More recently, Dafermos[5] and Ericksen[6] present detailed theoretical investigations of these experiments. Also, Leslie[7] shows that the theory predicts similar behaviour in other situations.

A sample of cholesteric liquid crystal at rest between parallel plates commonly has a twisted or helical orientation pattern. In this arrangement, the orientation of the molecular axes is everywhere parallel to the solid surfaces, being constant in any plane parallel to the plates, but varying uniformly with distance normal to the surfaces. As one traverses the gap between the plates, the ends of the molecules therefore trace out a helix, whose pitch is a characteristic of a given cholesteric liquid crystal. Employing the theory of Frank[2] and ignoring the influence of boundaries, de Gennes[8] and Meyer[9] discuss the application of magnetic fields to this type of liquid crystal. When the field is perpendicular to the helical axes, de Gennes finds that the pitch increases with field strength until at a critical value the molecules align parallel to the field throughout the sample. The experiments of Durand *et al.*[10] and Meyer[11] confirm this behaviour. De Gennes[8] and Meyer[9] also discuss the case when the field is parallel to the helical axes, but in the absence of boundary constraints consider the possibility that the sample reorientate so that the field is again perpendicular to the helical axes. However, Meyer also discusses the possibility of distortion of the helical structure.

In this paper we consider a liquid crystal at rest between parallel plates exhibiting the twisted orientation pattern described above, and employing continuum theory discuss the application of a magnetic field perpendicular to the plates. For nematic liquid crystals, if one assumes that the solid surfaces control the orientation of the molecules in contact, it seems possible to obtain such a twisted orientation pattern from the uniform parallel orientation pattern mentioned earlier simply by rotating one plate through an angle about its normal. For this type of liquid crystal, provided that the twist is not too large, our analysis suggests that no distortion occurs until one exceeds a critical field strength, which depends upon the amount of twist and the distance between the plates. With data from the experiments described above, one can predict this critical value, and this appears to offer a useful test for consistency between theory and experiment.

For cholesteric liquid crystals, we assume that the molecules align parallel to the boundaries such that the surfaces exert no normal couple stress upon the liquid crystal. Leslie(12,13) makes this assumption for this type of liquid crystal in other problems, and his predictions based upon it appear compatible with observations. As for the nematic case, our analysis suggests that no distortion of the orientation pattern can occur until the magnetic field strength exceeds a critical value. However, depending upon the relative magnitudes of two material parameters in the Frank energy, distortion may occur only when the distance between the plates is sufficiently small. Assuming that theory and experiment are consistent, it seems possible to obtain information regarding material parameters from these experiments.

2. Continuum Theory

This section presents a brief summary of the equations given by Ericksen(14) and Leslie(12,15) to describe static, isothermal behaviour of liquid crystals. It is convenient to choose a set of right-handed Cartesian axes, and to employ Cartesian tensor notation.

As is commonly done, one describes the orientation of the molecular axis by a unit vector d_i, and assumes that d_i and $-d_i$ are physically indistinguishable. The relevant equations are a balance of forces

$$t_{ij,j} + F_i = 0, \tag{2.1}$$

t_{ij} representing the stress tensor, and F_i the body force per unit volume, and a further balance law

$$s_{ij,j} + g_i + G_i = 0, \tag{2.2}$$

where s_{ij} is an orientation stress tensor, and g_i and G_i are intrinsic and extrinsic orientation body forces per unit volume respectively. Associated with the orientation stress tensor is a couple stress tensor l_{ij} given by

$$l_{ij} = e_{ipq} d_p s_{qj} . \tag{2.3}$$

The constitutive equations for the stress tensors and the intrinsic orientation body force are

$$t_{ij} = -p\delta_{ij} - \frac{\partial W}{\partial d_{k,j}} d_{k,i} + \alpha\, e_{jkp} (d_p d_i)_{,k} , \tag{2.4}$$

$$s_{ij} = d_i \beta_j + \frac{\partial W}{\partial d_{i,j}} + \alpha\, e_{ijk} d_k , \tag{2.5}$$

$$g_i = \gamma\, d_i - (d_i \beta_j)_{,j} - \frac{\partial W}{\partial d_i} - \alpha\, e_{ijk} d_{k,j} . \tag{2.6}$$

The undetermined pressure p in Eq. (2.4) arises on account of the assumed incompressibility, and similarly the scalar γ and the vector β_i in Eqs. (2.5) and (2.6) stem from the constraint that the vector d_i has fixed magnitude. Also, the function W is the Helmholtz free energy per unit volume, and depends only upon the vector d_i and its gradients. Here, we adopt the form due to Frank,[2]

$$\begin{aligned} 2W = {} & \alpha_1 (d_{i,i})^2 + \alpha_2 (\tau + d_i\, e_{ijk}\, d_{k,j})^2 \\ & + \alpha_3 d_i d_j d_{k,i} d_{k,j} + (\alpha_2 + \alpha_4)\, [d_{i,j} d_{j,i} - (d_{i,i})^2]. \end{aligned} \tag{2.7}$$

In the present context, the coefficients in the above expression and α are constants. For nematic liquid crystals, the coefficients α and τ are both zero.

In this paper, we consider body forces arising from an applied magnetic field H_i and gravity. Consequently, accepting the estimates of Ericksen[14]

$$F_i = -\chi_{,i} + \{(\nu_1 - \nu_2)\, H_j d_j d_k + \nu_2 H_k\}\, H_{k,i} , \tag{2.8}$$

$$G_i = (\nu_1 - \nu_2)\, H_j d_j H_i , \tag{2.9}$$

where χ is the gravitational potential, and ν_1 and ν_2 are the magnetic susceptibilities parallel and perpendicular to the molecular axis respectively. As in earlier work, we assume that ν_1 and ν_2 are positive constants, and that the former is the larger.

With the aid of Eqs. (2.5), (2.6) and (2.9), Eq. (2.2) becomes

$$\left(\frac{\partial W}{\partial d_{i,j}}\right)_{,j} - \frac{\partial W}{\partial d_i} + \gamma d_i + (\nu_1 - \nu_2) H_j d_j H_i = 0. \tag{2.10}$$

Also, when the external body forces take the form (2.8) and (2.9), it is straightforward to show that Eq. (2.1) integrates with the aid of Eq. (2.10) to yield

$$p = p_0 - \chi - W + \tfrac{1}{2}[(\nu_1 - \nu_2)(H_j d_j)^2 + \nu_2 H_j H_j], \tag{2.11}$$

where p_0 is an arbitrary constant. Consequently, below we seek solutions of Eq. (2.10).

3. The Solution

Referred to a set of right-handed Cartesian axes, consider a solution of Eqs. (2.7) and (2.10) of the form

$$d_x = \cos\theta(z)\cos\phi(z), \quad d_y = \cos\theta(z)\sin\phi(z), \quad d_z = \sin\theta(z), \tag{3.1}$$

with

$$H_x = 0, \; H_y = 0, \; H_z = H, \tag{3.2}$$

where H is a constant. After some manipulation to eliminate the scalar γ the Eq. (2.10) reduces to

$$f(\theta)\frac{d^2\theta}{dz^2} + \tfrac{1}{2}\frac{d}{d\theta}f(\theta)\left(\frac{d\theta}{dz}\right)^2 - \tfrac{1}{2}\frac{d}{d\theta}g(\theta)\left(\frac{d\phi}{dz}\right)^2 - 2\alpha_2\tau\sin\theta\cos\theta\frac{d\phi}{dz} + \nu H^2\sin\theta\cos\theta = 0, \tag{3.3}$$

and

$$g(\theta)\frac{d^2\phi}{dz^2} + \frac{d}{d\theta}g(\theta)\frac{d\theta}{dz}\frac{d\phi}{dz} + 2\alpha_2\tau\sin\theta\cos\theta\frac{d\theta}{dz} = 0, \tag{3.4}$$

in which

$$f(\theta) = \alpha_1\cos^2\theta + \alpha_3\sin^2\theta, \quad g(\theta) = (\alpha_2\cos^2\theta + \alpha_3\sin^2\theta)\cos^2\theta, \tag{3.5}$$

and

$$\nu = \nu_1 - \nu_2. \tag{3.6}$$

Eq. (3.4) promptly integrates to yield

$$g(\theta)\frac{d\phi}{dz} - \alpha_2\tau\cos^2\theta = k, \tag{3.7}$$

where k is an arbitrary constant. Below, we examine solutions in which both θ and ϕ vary. For these, multiply Eq. (3.3) by the derivative of θ and Eq. (3.4) by the derivative of ϕ, add and integrate to obtain

$$f(\theta)\left(\frac{d\theta}{dz}\right)^2+g(\theta)\left(\frac{d\phi}{dz}\right)^2+\nu H^2\sin^2\theta=c, \tag{3.8}$$

where c is an arbitrary constant. From Eqs. (3.7) and (3.8), it is straightforward to determine derivatives of θ and ϕ as functions of θ, and the solution follows readily by further integration. However, since the boundary conditions considered differ for nematic and cholesteric liquid crystals, it is necessary to discuss these cases separately.

4. Nematic Liquid Crystals

For this type of liquid crystal, material symmetries require that in Eqs. (2.4)–(2.7)

$$\alpha=0,\quad \tau=0. \tag{4.1}$$

Also, the argument of Ericksen[16] leads one to impose the conditions

$$\alpha_1>0,\quad \alpha_2>0,\quad \alpha_3>0. \tag{4.2}$$

We consider a layer of liquid crystal lying between parallel plates which coincide with the planes $z=0$ and $z=2l$. As remarked earlier, there is evidence that a solid surface imposes a particular orientation upon the molecules of a nematic liquid crystal immediately adjacent to it. Further, this is controllable to some degree by the prior treatment given to the surface. Consequently, it appears physically relevant to discuss the boundary conditions

$$\theta(0)=0,\ \theta(2l)=0,\ \phi(0)=-\phi_0,\ \phi(2l)=\phi_0, \tag{4.3}$$

where ϕ_0 is an arbitrary positive constant. In the absence of a reasonable alternative, we assume that the magnetic field does not alter the orientation at the solid interface. However, Saupe[4] suggests that this may be open to question on occasion.

When the magnetic field is absent, one solution of Eqs. (3.3) and (3.4) which meets the boundary conditions (4.3) is

$$\theta=0,\quad \phi=\phi_0(z-l)/l. \tag{4.4}$$

Provided that ϕ_0 is sufficiently large, it appears to be possible to construct other acceptable solutions in which θ varies. For the present, we disregard these and assume that the solution (4.4) represents the initial orientation pattern. This presumably places some restriction on the range of the parameter ϕ_0.

When the field is present, the twisted orientation pattern (4.4) remains a solution of Eqs. (3.3) and (3.4), but it is natural to examine another possibility. Consider solutions of the type

$$\theta(z) = \theta(2l - z), \quad 0 \leqslant z \leqslant l, \tag{4.5}$$

with

$$\theta(l) = \theta_m, \quad \frac{d}{dz}\theta(l) = 0, \tag{4.6}$$

θ_m being a constant to be determined. On account of the symmetry of the problem, one may choose θ_m positive without loss of generality. In this event, Eqs. (3.7) and (3.8) at once yield

$$g(\theta)\frac{d\phi}{dz} = k, \tag{4.7}$$

and

$$f(\theta)\left(\frac{d\theta}{dz}\right)^2 = \nu H^2\left(\sin^2\theta_m - \sin^2\theta\right) + k^2\left(\frac{1}{g(\theta_m)} - \frac{1}{g(\theta)}\right). \tag{4.8}$$

From Eq. (4.7), it follows that

$$\phi(z) = -\phi(2l - z), \quad 0 \leqslant z \leqslant l, \tag{4.9}$$

and in particular that

$$\phi(l) = 0. \tag{4.10}$$

The relevant solution is therefore given by

$$z = \int_0^\theta \left[\frac{f(\psi)}{\nu H^2(\sin^2\theta_m - \sin^2\psi) + k^2(1/g(\theta_m) - 1/g(\psi))}\right]^{1/2} d\psi, \quad 0 \leqslant z \leqslant l, \tag{4.11}$$

and

$$\phi = -\phi_0 + \int_0^\theta \left[\frac{f(\psi)}{\nu H^2(\sin^2\theta_m - \sin^2\psi) + k^2(1/g(\theta_m) - 1/g(\psi))}\right]^{1/2} \frac{k\,d\psi}{g(\psi)}, \quad 0 \leqslant z \leqslant l, \tag{4.12}$$

provided that the constants θ_m and k satisfy

$$l = \int_0^{\theta_m} \left[\frac{f(\theta)}{\nu H^2(\sin^2\theta_m - \sin^2\theta) + k^2(1/g(\theta_m) - 1/g(\theta))}\right]^{1/2} d\theta, \tag{4.13}$$

and

$$\phi_0 = \int_0^{\theta_m} \left[\frac{f(\theta)}{\nu H^2(\sin^2\theta_m - \sin^2\theta) + k^2(1/g(\theta_m) - 1/g(\theta))}\right]^{1/2} \frac{k\,d\theta}{g(\theta)}. \quad (4.14)$$

The last two equations therefore determine the parameters θ_m and k for a given field strength H. Alternatively, one may regard them as equations which give the field strength H and the parameter k as functions of θ_m.

If one makes a change of variable

$$\sin\lambda = \frac{\sin\theta}{\sin\theta_m}, \quad (4.15)$$

Eqs. (4.13) and (4.14) become

$$l = \int_0^{\pi/2} \left[\frac{f(\theta)}{\nu H^2 - k^2 F(\theta, \theta_m)/g(\theta)\,g(\theta_m)}\right]^{1/2} \frac{d\lambda}{\cos\theta}, \quad (4.16)$$

$$\phi_0 = \int_0^{\pi/2} \left[\frac{f(\theta)}{\nu H^2 - k^2 F(\theta, \theta_m)/g(\theta)\,g(\theta_m)}\right]^{1/2} \frac{k\,d\lambda}{\cos\theta\, g(\theta)}, \quad (4.17)$$

where

$$F(\theta, \theta_m) = \frac{[g(\theta_m) - g(\theta)]}{(\sin^2\theta_m - \sin^2\theta)}$$
$$= \alpha_3 - 2\alpha_2 - (\alpha_3 - \alpha_2)(\sin^2\theta + \sin^2\theta_m). \quad (4.18)$$

From these, it follows that

$$\lim_{\theta_m \to 0} k = \alpha_2 \phi_0/l, \quad \lim_{\theta_m \to 0} \nu l^2 H^2 = \alpha_1(\pi/2)^2 + (\alpha_3 - 2\alpha_2)\phi_0^2, \quad (4.19)$$

the latter assuming that

$$\text{either} \quad \alpha_3 \geqslant 2\alpha_2, \quad \text{or} \quad \phi_0^2 \leqslant \alpha_1 \pi^2/4(2\alpha_2 - \alpha_3). \quad (4.20)$$

With the notation

$$\beta = \sin^2\theta_m, \quad (4.21)$$

differentiation of expressions (4.16) and (4.17) leads to

$$2l\left(\frac{dk}{d\beta}\right)_{\beta=0} = (\alpha_3 - 2\alpha_2)\phi_0, \quad (4.22)$$

$$2\nu l^2\left(\frac{dH^2}{d\beta}\right)_{\beta=0} = \alpha_3(\pi/2)^2 - (\alpha_3^2 - \alpha_3\alpha_2 + \alpha_2^2)\phi_0^2/\alpha_2. \quad (4.23)$$

Consequently, if the condition (4.20) holds, and also

$$\phi_0^2 < \alpha_3\alpha_2\pi^2/4(\alpha_3^2 - \alpha_3\alpha_2 + \alpha_2^2), \quad (4.24)$$

one has the result that

$$\frac{dH}{(d\beta)_{\beta=0}} > 0. \tag{4.25}$$

The conditions (4.20) and (4.24) are therefore necessary to ensure that the field strength increases monotonically with θ_m in the interval zero to $\pi/2$. However, it is beyond the scope of the present investigation to determine sufficient conditions for this.

Given the existence of more than one solution, following Dafermos[5] we select on stability grounds that which minimizes the energy function

$$\epsilon = \int_V [W - \tfrac{1}{2}\{(\nu_1 - \nu_2)(H_i d_i)^2 + \nu_2 H_i H_i\}]\, dV, \tag{4.26}$$

where V is the volume of liquid crystal. If one denotes by ϵ_0 the value of the above integral when the solution (4.4) occurs, and introduces an energy difference

$$\Delta = \epsilon - \epsilon_0, \tag{4.27}$$

Eqs. (2.7), (3.1), (3.2), (3.5), (3.6) and (4.1) yield

$$\Delta = \tfrac{1}{2}A \int_0^{2l} \left[f(\theta)\left(\frac{d\theta}{dz}\right)^2 + g(\theta)\left(\frac{d\phi}{dz}\right)^2 - \nu H^2 \sin^2\theta - \alpha_2 \phi_0{}^2/l^2 \right] dz, \tag{4.28}$$

where A is the area of the plates. Hence, with the aid of Eqs. (4.5), (4.7) and (4.8), one obtains

$$\Delta = A \int_0^l [\nu H^2(\sin^2\theta_m - 2\sin^2\theta) + k^2/g(\theta_m) - \alpha_2 \phi_0{}^2/l^2]\, dz, \tag{4.29}$$

and changes of variable employing Eqs. (4.8) and (4.15) lead to

$$\Delta = A \int_0^{\pi/2} \frac{[f(\theta)]^{1/2}\,[\nu H^2 \sin^2\theta_m \cos 2\lambda + k^2/g(\theta_m) - \alpha_2 \phi_0{}^2/l^2]}{\cos\theta\,[\nu H^2 - k^2 F(\theta, \theta_m)/g(\theta)\, g(\theta_m)]^{1/2}}\, d\lambda. \tag{4.30}$$

By differentiation of this expression and the relations (4.16) and (4.17), one can show that

$$\left(\frac{d\Delta}{d\beta}\right)_{\beta=0} = 0,\ 4l\left(\frac{d^2\Delta}{d\beta^2}\right)_{\beta=0} = -A[\alpha_3(\pi/2)^2 - (\alpha_3^2 - \alpha_3\alpha_2 + \alpha_2^2)\phi_0{}^2/\alpha_2]. \tag{4.31}$$

Therefore, provided that the conditions (4.20) and (4.24) hold,

$$\epsilon > \epsilon_0 \tag{4.32}$$

for values of θ_m in some neighbourhood of zero.

The above analysis points to the existence of a critical magnetic field strength H_c, given by

$$\nu l^2 H_c^2 = \alpha_1 (\pi/2)^2 + (\alpha_3 - 2\alpha_2)\phi_0^2, \tag{4.33}$$

provided that the angle ϕ_0 is not too large. For field strengths below this value, the twisted orientation pattern occurs, and above this value there is distortion of this orientation pattern. As Saupe[4] demonstrates, it is possible to estimate the three Frank constants appearing in the expression (4.33) by means of the experiments mentioned earlier, allowing one to predict the critical field strength for the present situation. This seems to offer a worthwhile opportunity to compare theory and experiment. Further, it could provide a means of assessing the range of applicability of the Frank free energy. As Ericksen[16] discusses, this energy essentially represents an expansion about a uniform orientation pattern, and therefore should be appropriate to describe small departures from such an orientation pattern, this being the case in the experiments employed by Saupe.[4] Here, however, the initial orientation pattern is non-uniform, the degree of non-uniformity being given by the ratio ϕ_0/l. Consequently, the extent of agreement between the relationship (4.33) and experiment as the distance between the plates decreases should give some indication of the range of validity of this form of energy.

5. Cholesteric Liquid Crystals

For this type of liquid crystal, the static form of the theory of Leslie[12] differs from that due to Ericksen,[14] the terms with coefficient α in Eqs. (2.4)–(2.6) appearing only in the former. These arise in a natural way in Leslie's derivation, but the need to include them in the theory remains to be established. Since their absence leads to some simplification, we initially omit them. In this event, application of Ericksen's[16] reasoning to the commonly observed twisted orientation pattern leads to

$$\alpha_1 > 0, \quad \alpha_2 > 0, \quad \alpha_3 > 0, \quad \alpha_2 + \alpha_4 = 0, \quad \tau = \frac{2\pi}{\zeta}, \tag{5.1}$$

where ζ denotes the pitch of the helix.

Here, the liquid crystal again lies between parallel plates coincident with the planes $z = 0$ and $z = 2l$. On this occasion, however, the boundary conditions for the molecular orientation are less clear, and would seem to depend upon the nature of the material forming the plates. One reasonable proposition follows from the common occurrence of the characteristic twisted orientation pattern, namely that at a solid boundary the molecular axes align parallel to the surface such that the normal component of couple stress is zero. Leslie[12,13] employs this condition, and his conclusions based upon it appear consistent with observations. From Eqs. (2.3), (2.5), (2.7), (3.1) and (3.5), the relevant component of couple stress is

$$l_{zz} = g(\theta)\frac{d\phi}{dz} - \alpha_2 \tau \cos^2\theta, \tag{5.2}$$

and therefore the boundary conditions considered below are

$$\theta(0) = 0, \quad \theta(2l) = 0, \quad \frac{d}{dz}\phi(0) = \tau, \quad \frac{d}{dz}\phi(2l) = \tau. \tag{5.3}$$

One solution of Eqs. (3.3) and (3.4), compatible with the boundary conditions (5.3), is the characteristic twisted orientation pattern,

$$\theta = 0, \quad \phi = \phi_1 + \tau z, \tag{5.4}$$

where ϕ_1 is an arbitrary constant. When the magnetic field is absent, one can show that this is the only solution satisfying the conditions (5.3). However, when the field is present, another is possible.

As in the previous section, consider solutions in which

$$\theta(z) = \theta(2l - z), \quad 0 \leqslant z \leqslant l, \tag{5.5}$$

with

$$\theta(l) = \theta_m, \quad \frac{d}{dz}\theta(l) = 0, \tag{5.6}$$

θ_m again being a positive constant. In view of the boundary conditions (5.3), Eq. (3.7) takes the form

$$\frac{d\phi}{dz} = \frac{\alpha_2 \tau}{h(\theta)}, \quad h(\theta) = \alpha_2 \cos^2\theta + \alpha_3 \sin^2\theta. \tag{5.7}$$

With this result and the conditions (5.6), Eq. (3.8) becomes

$$f(\theta)\left(\frac{d\theta}{dz}\right)^2 = \nu H^2(\sin^2\theta_m - \sin^2\theta) + \alpha_2^2\tau^2\left(\frac{\cos^2\theta_m}{h(\theta_m)} - \frac{\cos^2\theta}{h(\theta)}\right). \tag{5.8}$$

The appropriate solution is therefore

$$z = \int_0^\theta \left[\frac{f(\psi)}{\nu H^2(\sin^2\theta_m - \sin^2\psi) + \alpha_2^2\tau^2(\cos^2\theta_m/h(\theta_m) - \cos^2\psi/h(\psi))}\right]^{1/2} d\psi, \quad 0 \leqslant z \leqslant l, \tag{5.9}$$

provided that the constant θ_m satisfies

$$l = \int_0^{\theta_m} \left[\frac{f(\theta)}{\nu H^2(\sin^2\theta_m - \sin^2\theta) + \alpha_2^2\tau^2(\cos^2\theta_m/h(\theta_m) - \cos^2\theta/h(\theta))}\right]^{1/2} d\theta, \tag{5.10}$$

this last equation relating θ_m and the field strength H.

The substitution (4.15) reduces Eq. (5.10) to

$$l = \int_0^{\pi/2} \left[\frac{f(\theta)}{\nu H^2 - \alpha_3\alpha_2^2\tau^2/h(\theta)\,h(\theta_m)}\right]^{1/2} \frac{d\lambda}{\cos\theta}, \tag{5.11}$$

from which one readily obtains

$$\lim_{\theta_m\to 0} \nu H^2 = \alpha_1\left(\frac{\pi}{2l}\right)^2 + \alpha_3\tau^2. \tag{5.12}$$

Also, with the notation (4.21), differentiation of the expression (5.11) leads to

$$2\nu\left(\frac{dH^2}{d\beta}\right)_{\beta=0} = \alpha_3\,[(\pi/2l)^2 + 3(\alpha_2 - \alpha_3)\,\tau^2/\alpha_2], \tag{5.13}$$

and therefore

$$\left(\frac{dH}{d\beta}\right)_{\beta=0} > 0, \tag{5.14}$$

provided that,

$$\text{either} \qquad \alpha_2 \geqslant \alpha_3, \qquad \text{or} \qquad l^2 < \frac{\alpha_2\pi^2}{12(\alpha_3 - \alpha_2)\tau^2}. \tag{5.15}$$

To discriminate between the solutions, we examine their energies as defined by Eq. (4.26). Denoting that of the solution (5.4) by ϵ_0, and that of the second by ϵ, let

$$\Delta = \epsilon - \epsilon_0, \tag{5.16}$$

and Eqs. (2.7), (3.1), (3.2), (3.5) and (3.6) combine to yield

$$\Delta = \tfrac{1}{2}A\int_0^{2l}\left[f(\theta)\left(\frac{d\theta}{dz}\right)^2 + g(\theta)\left(\frac{d\phi}{dz}\right)^2 - 2\alpha_2\tau\cos^2\theta\frac{d\phi}{dz} + \alpha_2\tau^2 - \nu H^2\sin^2\theta\right]dz, \qquad (5.17)$$

A being the area of the plates. With the aid of Eqs. (5.5), (5.7) and (5.8), one obtains

$$\Delta = A\int_0^{l}\left[\nu H^2(\sin^2\theta_m - 2\sin^2\theta) + \alpha_2\tau^2\left\{\alpha_2\frac{\cos^2\theta_m}{h(\theta_m)} - 2\alpha_2\frac{\cos^2\theta}{h(\theta)} + 1\right\}\right]dz, \qquad (5.18)$$

and proceeding as before it follows that

$$\Delta = A\int_0^{\pi/2}\frac{\left[f(\theta)\right]^{1/2}\left[\nu H^2\sin^2\theta_m\cos 2\lambda + \alpha_2\tau^2\left\{\alpha_2\frac{\cos^2\theta_m}{h(\theta_m)} - 2\alpha_2\frac{\cos^2\theta}{h(\theta)} + 1\right\}\right]}{\cos\theta\,[\nu H^2 - \alpha_3\alpha_2^2\tau^2/h(\theta)\,h(\theta_m)]^{1/2}}\,d\lambda. \qquad (5.19)$$

Differentiation of this last expression leads to

$$\left(\frac{d\Delta}{d\beta}\right)_{\beta=0} = 0, \quad 4\left(\frac{d^2\Delta}{d\beta^2}\right)_{\beta=0} = -Al\alpha_3\left[\left(\frac{\pi}{2l}\right)^2 + 3(\alpha_2-\alpha_3)\frac{\tau^2}{\alpha_2}\right], \qquad (5.20)$$

and thus, if the condition (5.15) holds,

$$\epsilon < \epsilon_0 \qquad (5.21)$$

for values of θ_m in a neighbourhood of the origin.

For this type of liquid crystal, therefore, the above analysis points to the occurrence of a critical field strength H_c, given by

$$\nu H_c^2 = \alpha_1\left(\frac{\pi}{2l}\right)^2 + \alpha_3\tau^2, \qquad (5.22)$$

provided that the condition (5.15) is satisfied. For field strengths below this value, the magnetic field does not influence the twisted orientation pattern, but above this value distortion occurs. With the aid of the relations (5.1), the inequality in condition (5.15) is equivalent to

$$l < \left[\frac{\alpha_2}{3(\alpha_3-\alpha_2)}\right]^{1/2}\frac{\zeta}{4}. \qquad (5.23)$$

Consequently, if the Frank constant α_3 is larger than α_2, this requirement places a severe restriction on the distance between the plates. However, if α_2 is greater than α_3 measurements when l is large compared with the characteristic pitch ζ should provide an estimate of α_3.

When one retains the terms with coefficient α in Eqs. (2.4)–(2.6), the argument preceding the conditions (5.1) requires modification. In this case, Leslie[13] provides motivation for

$$\alpha_1 > 0, \quad \alpha_2 > 0, \quad \alpha_3 > 0, \quad \tau - \frac{\alpha}{\alpha_2} = 2\pi/\zeta, \tag{5.24}$$

ζ again being the pitch of the characteristic helical arrangement.

With the retention of these terms, the component of couple stress normal to the plates is

$$l_{zz} = g(\theta)\frac{d\phi}{dz} + (\alpha - \alpha_2\tau)\cos^2\theta, \tag{5.25}$$

and therefore it is necessary to replace the boundary conditions by

$$\theta(0) = 0, \quad \theta(2l) = 0, \quad \frac{d}{dz}\phi(0) = \tau - \frac{\alpha}{\alpha_2}, \quad \frac{d}{dz}\phi(2l) = \tau - \frac{\alpha}{\alpha_2}. \tag{5.26}$$

One solution of Eqs. (3.3) and (3.4) subject to the above boundary conditions is

$$\theta = 0, \quad \phi = \phi_1 + \left(\tau - \frac{\alpha}{\alpha_2}\right)z, \tag{5.27}$$

where ϕ_1 is an arbitrary constant. In addition, if one considers a solution of these equations satisfying conditions (5.5), (5.6) and (5.26), Eqs. (3.7) and (3.8) become

$$g(\theta)\frac{d\phi}{dz} = \alpha_2\tau\cos^2\theta - \alpha, \tag{5.28}$$

and

$$f(\theta)\left(\frac{d\theta}{dz}\right)^2 = \nu H^2(\sin^2\theta_m - \sin^2\theta) + \frac{(\alpha_2\tau\cos^2\theta_m - \alpha)^2}{g(\theta_m)} - \frac{(\alpha_2\tau\cos^2\theta - \alpha)^2}{g(\theta)}. \tag{5.29}$$

The solution is therefore

$$z = \int_0^\theta \left[\frac{f(\psi)}{\nu H^2(\sin^2\theta_m - \sin^2\psi) + \dfrac{(\alpha_2\tau\cos^2\theta_m - \alpha)^2}{g(\theta_m)} - \dfrac{(\alpha_2\tau\cos^2\psi - \alpha)^2}{g(\psi)}}\right]^{1/2} d\psi,$$

$$0 \leqslant z \leqslant l, \qquad (5.30)$$

provided that θ_m satisfies

$$l = \int_0^{\theta_m} \left[\frac{f(\theta)}{\nu H^2(\sin^2\theta_- - \sin^2\theta) + \dfrac{(\alpha_2\tau\cos^2\theta_m - \alpha)^2}{g(\theta_m)} - \dfrac{(\alpha_2\tau\cos^2\theta - \alpha)^2}{g(\theta)}}\right]^{1/2} d\theta. \qquad (5.31)$$

Here, one notes the possibility of a solution of this type when the field is absent.

With the substitution (4.15), the expression (5.31) becomes

$$l = \int_0^{\pi/2}\left[\frac{f(\theta)}{\nu H^2 - G(\theta,\theta_m)/g(\theta)\,g(\theta_m)}\right]^{1/2}\frac{d\lambda}{\cos\theta}, \qquad (5.32)$$

where

$$G(\theta,\theta_m) = [\alpha_3\alpha_2^2\tau^2 - 2\alpha_2\tau\alpha(\alpha_3-\alpha_2)]\cos^2\theta\cos^2\theta_m$$
$$+\alpha^2[(\alpha_3 - 2\alpha_2) + (\alpha_2-\alpha_3)(\sin^2\theta + \sin^2\theta_m)]. \qquad (5.33)$$

From Eq. (5.32), one readily finds that

$$\lim_{\theta_m\to 0}\nu H^2 = \alpha_1\left(\frac{\pi}{2l}\right)^2 + \alpha_3\left(\tau - \frac{\alpha}{\alpha_2}\right)^2 + 2\alpha\left(\tau - \frac{\alpha}{\alpha_2}\right), \qquad (5.34)$$

and that

$$2\alpha_2\nu\left(\frac{dH^2}{d\beta}\right)_{\beta=0} = \alpha_2\alpha_3\left(\frac{\pi}{2l}\right)^2 + 3\alpha_3(\alpha_2-\alpha_3)\left(\tau - \frac{\alpha}{\alpha_2}\right)^2$$
$$+ 6\alpha(\alpha_2-\alpha_3)\left(\tau - \frac{\alpha}{\alpha_2}\right) - 3\alpha^2. \qquad (5.35)$$

Consequently, the retention of the terms with coefficient α does not necessarily alter the nature of some of our earlier results. However, when they are present, the stability argument employed above appears to require modification, and it is beyond the scope of this paper to consider this question further. It is perhaps worth noting that the existence of a critical field strength given by Eq. (5.34) offers some prospect of obtaining information on the coefficient α.

Finally, as Ericksen[16] points out, the form of energy employed in this paper for cholesteric liquid crystals is open to criticism in that it is an expansion about a uniform orientation pattern rather than about the characteristic twisted orientation pattern. In a recent paper, Jenkins[17] proposes an energy for cholesteric liquid crystals which remedies this. However, predictions based on this energy have yet to be determined.

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